JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 20, 442-453 (1967)

# Stability and Convergence of Approximation Schemes*

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## 1. INTRODUCTION

The notion of stability is strongly intuitive. The many mathematical applications and points-of-view have failed to agree upon a universal definition, although the concept is deserving of such generalization. Desiring to consider the discrete approximation of continuous problems without becoming encumbered with an unwieldy notation and assumptions about mesh configurations, assignment of variables to meshes, and types of discretizations, we are led to consider well-posed problems on normed spaces and to suggest a parent concept (namely, uniform continuity) under which several notions of stability may be discussed.

That various stabilities described in connection with Liapunov theory, mechanics, and numerical analysis may be characterized as uniform continuities is shown. To illustrate the numerical case we develop the Lax-Richtmyer [1] theory of numerical stability for linear solution operators. This development also serves to motivate our study of the abstracted, nonlinear problem.

In section four it is seen that the extension of the numerical stability concept to nonlinear solution operators must involve the operator being approximated in contrast to the linear case in which the "domain of stability" may be translated to the origin. However, a formulation is given which will imply convergence, given consistency. Moreover, the linearized stability analysis commonly performed (following von Neumann and Richtmyer [2]) is seen to be a natural approximation of a criterion based on the numerical stability defined for the nonlinear problem. This analysis applies not only to the Lax-Richtmyer theory which it resembles, but to all initial value problems which

* This work was initially supported by the U. S. Army Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland. It arose as a consequence of conversations with Mr. R. C. Makino and Dr. J. H. Giese. The final form was achieved while the author was employed by Rensselaer Polytechnic Institute.

are well-posed on some normed or Banach space. In particular, the stability and convergence theorem for first-order ordinary differential equations is included (see Isaacson and Keller [3] which also contains a very suggestive discussion of "well-posedness" of numerical problems). Existence and uniqueness theorems for nonlinear partial differential equation initial/boundary-value problems may be found in Friedman [5] and Sobolev [6], for instance.

## 2. Stability as Uniform Continuity

Let us consider two topological spaces $X$ and $Y$ with $Y$ possessing a uniformity. Also, let $J$ represent any set and $S$ a mapping

$$S : X \times J \to Y.$$

If $x_0$ is in $X$, we shall say that $S$ is *stable with respect to* $(X, Y, J)$ *at* $x_0$ if and only if $S$ is continuous at $x_0$ uniformly over $J$. Graphically (Fig. 1), to any

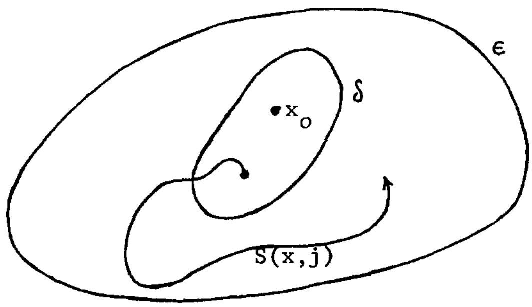


Fig. 1

neighborhood ($\epsilon$) of $x_0$ there corresponds a neighborhood ($\delta$) such that each trajectory initiated in the latter ($\delta$) is confined to the former:

$$\{S(x, j) \mid x \in \delta, j \in J\} \subset \epsilon.$$

To illustrate the validity of this use of the term "stability," let us consider a few slightly less general situations. We denote by $R$ the real numbers with their usual topology and uniformity and by $R(\geqslant 0)$ the subspace of non-negative reals.

*Situation 1.* Let $J = R(\geqslant 0)$ and $X = Y = R$. Interpreting $x_0$ as an initial value $y(0) = x_0$ of the solution $y(t)$ of a first-order differential equation, we set $S(x_0, t) = y(t)$, $t \in J$. The condition of uniform continuity requires that at all times $t \in J$, the effect of an initial disturbance will disappear as that disturbance is eliminated and, moreover, that this will occur comparably fast at each $t$.

*Situation 2.* Let $J = R(\geqslant 0)$, $X$ be a suitable subspace of $(R^n)^{R^m}$, for positive integers $m$ and $n$, and $Y = X$. The elements of $X$ are thus $n$-tuples of real-valued functions of $m$ independent, real variables. The notation $x(z_1,..., z_m) \in X$, $t \in J$, $S(x_0, t) = y(t, z_1,..., z_m)$, where $x$ and $y$ are $n$-tuples of dependent variables, is used. This occurs with a partial differential equation in $n$ variables, dependent upon $m$ independent variables $z_1,..., z_m$ and on time $t$. The operator $S$ may be regarded as the solution operator of the initial value problem $y(0, z_1,..., z_m) = x_0(z_1,..., z_m)$. Thus stability requires that at each point $(z_1,..., z_m)$ the consequences of disturbances of the initial values disappear, uniformly in time, as the magnitude of the initial disturbances is decreased toward zero. Variations on this may be obtained by assigning some or all of the space variables $z_1,..., z_m$ to $J$.

*Situation 3.* In the study of partial differential equations one may be interested in local stability, rather than the stability-in-the-large of the previous case; that is, an observer at a fixed position in space may be unconcerned about the solution elsewhere. Stability for this observer is formulated by taking $J = R(\geqslant 0)$, $X \subset (R^n)^{R^m}$ and $Y = R^n$ with

$$S(x_0(z_1,..., z_m), t) = y(t, a_1,..., a_m)$$

in the notation of Situation 2. The $m$-tuple $(a_1,..., a_m)$ represents the observation point in $R^m$.

*Situation 4.* Let $J = R^2$, $X = Y = R^n$, and $(t, \tau) \in J$. If

$$S(x_0, t, \tau) = y(t, \tau)$$

is regarded as the solution operator on a phase space $X$ of a system of ordinary differential equations with $y(\tau, \tau) = x_0$ as initial value, then stability is precisely the uniform stability of Lefschetz [12].

*Situation 5.* Let $J \subset R^k$, $X = R^{n+1}$, $Y = R^n$. Taking $x_0$ in $R^n$, $(x_0, t)$ in $X$, $\alpha$ in $J$, we regard $S$, given by $S(x_0, t, \alpha) = y(t, \alpha)$, as the solution operator of a system of differential equations, dependent on $\alpha$ and having initial value $y(0, \alpha) = x_0$. The stability of $S$ is a "structural" stability with respect to the $k$ real parameters composing $\alpha$.

The notion of practical stability mentioned by LaSalle and Lefschetz [7] is equivalent to stability at all points throughout some suitably large neighborhood of $x_0$. Asymptotic stability requires that $J$ possess a topology so that the effects of a disturbance may "die out" as a limit is approached in $J$. In most cases, asymptotic stability of an operator $S : X \times J \to Y$ may be described as the stability with respect to $(X \times J_1, Y^2, J_2)$ of the operator $T : X^2 \times J_1 \times J_2 \to Y^2$ at $(x, t) = (x_0, t_0)$ when $T$ is defined by

$$\begin{aligned} T(x, \xi, t, \tau) &= (S(x, \tau), S(\xi, t)), & t \neq t_0 \\ &= (S(x, \tau), S(x_0, t_0)), & t = t_0, \end{aligned}$$

with $t$ in $J_1 = J$, $\tau$ in $J_2 = J$, for each $\xi$ in some neighborhood $N$ of $x_0$. Here $S(x_0, t_0)$ represents the equilibrium state to which all trajectories initiated in $N$ tend as $t \to t_0$.

## 3. Numerical Stability

In [4] Courant *et al.* demonstrated conditions under which solutions of difference equations on a family of meshes converge to the solution of a partial differential equation as the distances between neighboring mesh points is decreased to zero. This work has inspired the use of a stability which requires continuity, uniform over a spectrum of meshes becoming refined without limit as well as over the range of the index of iteration or time. Lax and Richtmyer [1] have given this stability an operator-theoretic definition. Considerable structure is needed to develop the definition and to relate it to other stability concepts.

Let us attribute to $(R^n)^{R^m}$ the usual vector space structure and direct our attention to a normed subspace which we denote by $X_1$. The class of sets $M(j), j = 1, 2, \ldots$, whose members each consist of finitely many points of $R^m$ is said to approximate a subset $D \subset R^m$ provided that to each $p$ in $D$, there are $p_j$ in $M(j)$ such that $p_j \to p$ as $j \to \infty$.

For $f$ in $(R^n)^{M(j)}$ we let $f^e$ represent an extension of $f$ to an element of $X_1$. We require that there be a constant $K(j) > 0$ such that

$$[K(j)]^{-1} \|f\|_j \leqslant \|f^e\| \leqslant K(j) \|f\|_j,$$

where $\|f\|_j$ is a norm on $(R^n)^{M(j)}$. If $C$ maps $(R^n)^{M(j)}$ to itself, then a map $C^e$ of $X_1$ to itself may be defined by $C^e x = (Cf)^e$, where $f$ is the restriction of $x$ to $M(j)$. Consequently, if we are dealing with a linear extension ($f$ to $f^e$) and $C$ is linear, so is $C^e$. Moreover, every $C^e$ maps $X_1$ to itself and $C^e$ is a continuous map of $X$ whenever $C$ is continuous on $(R^n)^{M(j)}$.

This structure allows us to regard the solution operators for difference equation problems on many meshes $M(j)$ in $R^m$ as operating on a common domain $X_1$ and to discuss approximation of the difference equation solutions to one another as well as to the solutions of problems on a continuum $D$.

Now, as in [1], a definition of stability may be given for a difference scheme used to construct a linear difference operator $C(j, \tau, t)$, which carries the state $f$ in $(R^n)^{M(j)}$ at time $\tau$ to state $C(j, \tau, t)f$ at time $t$ (taking into account such boundary conditions as may be demanded). Computation is regarded as carried out on a sequence of time intervals to an elapsed time $t'$ in an interval $0 \leqslant t' \leqslant T < \infty$. Moreover, the computation is imagined to be carried out on each of a family of sequences of time intervals, denoted by $\{\Delta_{ij}t \mid 1 \leqslant i, j < \infty\}$, satisfying

$$\Delta_{ij}t \geqslant 0, \qquad \text{for all} \qquad i, j$$

$$\Delta_{ij}t \to 0, \qquad \text{as} \qquad j \to \infty \qquad \text{for each} \qquad i$$

$$t_{jk} = \sum_{i=1}^{k} \Delta_{ij}t \leqslant T, \qquad \text{for all} \qquad j, k \geqslant 1.$$

As a convenience we let $t_{j0} = 0$. This family of sequences is denoted hereafter by $W$. The set of positive integers is denoted by $J_0$.

The difference scheme (represented by $C$ or $C^e$) is said to be CFL-stable (numerically stable) on $\{\Delta_{ij}t\}$ in $W$, if the set of linear operators

$$\left\{\prod_{i=0}^{k-1} [C(j, t_{ji}, t_{ji+1})]^e \mid k, j \text{ in } J_0\right\}$$

is uniformly bounded. This is equivalent to requiring that

$$S : X_1 \times J_0 \times J_0 \to X_1,$$

defined by

$$S(x, j, k) = \left\{\prod_{i=0}^{k-1} [C(j, t_{ji}, t_{ji+1})]^e\right\} x$$

be continuous at $0$ in $X_1$ uniformly over $j, k \in J_0$. This would suggest a similar definition for nonlinear $C$.

## 4. Consistency and Convergence

Let $X$ be a normed space and let $W$ be the family of sequences defined in the previous section. Picking an arbitrary $t'$ in $0 \leqslant t' \leqslant T < \infty$ we will restrict ourselves to $t_{jk}$ with $k \leqslant k(j)$, an integer-valued function of $j$ having the property that $t_{jk(j)} \to t'$ as $j \to \infty$.

We consider mappings $U(w, j, h, k)$ and $V(w, j, h, k)$ to be defined for $w$ in $W$, $j \geqslant 0$, and $0 \leqslant h \leqslant k \leqslant k(j)$ on a portion of $X$ to $X$. These are the solution operators of the approximated and approximating initial value problems (with such boundary values as occur accounted for) which advance solutions from $t = t_{jh}$ to $t = t_{jk}$.

That we consider such solution operators necessitates the assumption that both approximated and approximating initial value problems be well-posed; that is, the problems have unique solutions, continuously dependent upon their initial values lying in some set $N_0 \subset X$. Since we want to focus on a particular initial value $x_0$ in $X$, we will assume $N_0$ to be a neighborhood of $x_0$. The uniqueness enforces the following relations:

$$U(w, j, k_1, k_2)\, U(w, j, k_0, k_1) = U(w, j, k_0, k_2)$$

and

$$U(w, j, k, k) = I, \qquad \text{the identity map,}$$

and similar relations for $V$. Thus with $U(w, j, 0, k)$ defined on $N_0$, $U(w, j, h, k)$ will be defined on $U(w, j, 0, h)\, N_0$. Generally difference equation solution operators are not nearly so restricted. Thus we will assume with no great loss of generality that for all $h$, $k$, $V(w, j, h, k)$ is defined on a neighborhood $N_1$ of the trajectories generated by $U$:

$$N_1 = \bigcup_{0 \leqslant j} \Big\{ \bigcup_{0 \leqslant h \leqslant k(j)} [U(w, j, 0, h)\, N_0] \Big\}.$$

Our operators exhibit $w$ and $j$ dependence because difference approximations commonly do so and the approximated as well as the approximating problem is potentially a difference, as well as a differential, problem. Having made this observation, we hereafter take the $w$-dependence for granted and cease to show $w$ as an argument.

The classic relation between $U$ and $V$ is called consistency. We will define *$V$ is consistent with $U$ on $w$* to mean that for each $x$ in some dense subset $N_0'$ of $N_0$

$$(\Delta_{h+1,j} t)^{-1} \,\|\, V(j, h, h+1)\, U(j, 0, h)\, x - U(j, h, h+1)\, U(j, 0, h)\, x \,\| \to 0$$

uniformly over $h$ in $0 \leqslant h \leqslant k(j)$ as $j \to \infty$.

As a first case let us return to the problem of linear solution operators $U$ and $V$ which have arisen in connection with homogeneous initial value differential and difference problems, respectively. Then $U$ and $V$, like the operator $C$ of the previous section, will depend only on the time interval $[\tau, t]$ through which the solutions are to be advanced.

THEOREM—LAX AND RICHTMYER [1]. *Let $X$ be a Banach space. If $U$ and $V$ are consistent linear mappings, then $V$ is CFL-stable in the sense of Section 3 if and only if*

$$\| U(j, 0, k(j))\, x - V(j, 0, k(j))\, x \| \to 0$$

*as $j \to \infty$ for all $x$ in $N_0$.*

In view of this theorem it would appear desirable to show that given consistency, stability, and convergence are equivalent in a nonlinear setting. However, that is not to be our pleasure here. Only in linear problems can we expect that adherence to a stability criterion, independent of the solution being approximated, by the approximating difference scheme will be sufficient to guarantee convergence. The coupling of $U$ and $V$ must go beyond the consistency criterion and influence the domain of stability of $V$. Of course, less than sharp results may appear uncoupled by demanding an unnecessarily global stability.

The mapping $V$ will be called a *CFL-stable approximation of $U$ at $x$* in $N_0$ if $V(j, h, k)$ is continuous at $U(j, 0, h)\, x$ and if this continuity is uniform over $0 \leqslant j$, $0 \leqslant h \leqslant k \leqslant k(j)$. In other words, if for each $\epsilon > 0$ there is a $\delta > 0$, independent of $j$, $h$, $k$ obeying $0 \leqslant j$, $0 \leqslant h \leqslant k \leqslant k(j)$, such that

$$\| V(j, h, k)\, x_1 - V(j, h, k)\, U(j, 0, h)\, x \| < \epsilon$$

whenever $\| x_1 - U(j, 0, h)\, x \| < \delta$, then $V$ is a CFL-stable approximation of $U$ at $x$. This is not a stability which fits into the mold prescribed in section two because of the involvement of $U(j, 0, h)$.

At this point a conjecture will be offered: if $V$ is consistent with $U$ on $w$ and $U(j, 0, h)$ is continuous on $N_0$ uniformly over $0 \leqslant j$, $0 \leqslant h \leqslant k(j)$, then $V$ is a CFL-stable approximation of $U$ at $x_0$ in $N_0$ if and only if for each $x$ in $U(j, 0, h)\, N_0$ and each $h$ in $0 \leqslant h \leqslant k(j)$

$$\| V(j, h, k(j))\, x - U(j, h, k(j))\, x \| \to 0$$

as $j \to \infty$. If the motivation for this were not so contorted that a manageable explanation were available, then it might be offered as a theorem.

The mapping $V$ is a CFL-stable approximation of $U$ at $x$ if there is a Lipschitzian constant $L(x)$, independent of $j$, $h$, $k$, making the inequality

$$\| V(j, h, k)\, x_1 - V(j, h, k)\, U(j, 0, h)\, x \| \leqslant L(x) \, \| x_1 - U(j, 0, h)\, x \| \tag{1}$$

valid for all $x_1$ within some positive radius of $U(j, 0, h)\, x$. In the case of a linear $V$ this reduces to the uniform boundedness demanded by the Lax theory. We are now ready for

THEOREM 1. *Let $V$ be consistent with $U$ on $w$ in $W$, let $k = k(j)$, and let $V$ obey the Lipschitzian condition (1) at each $x$ in $N_0$. Then*

$$\| V(j, 0, k)\, x - U(j, 0, k)\, x \|$$

*converges to zero for all $x$ in $N_0'$ as $j \to \infty$.*

PROOF. Observe

$$V(j, 0, k) - U(j, 0, k) = \sum_{h=0}^{k-1} [V(j, h, k)\, U(j, 0, h) - V(j, h+1, k)\, U(j, 0, h+1)].$$

Thus

$$\| V(j, 0, k)\, x - U(j, 0, k)\, x \|$$

$$\leqslant \sum_{h=0}^{k-1} \| V(j, h+1, k)\, V(j, h, h+1)\, U(j, 0, h)\, x - V(j, h+1, k)\, U(j, 0, h+1)\, x \|$$

$$\leqslant \sum_{h=0}^{k-1} L(x) \| V(j, h, h+1)\, U(j, 0, h)\, x - U(j, h, h+1)\, U(j, 0, h)\, x \|$$

$$= L(x) \sum_{h=0}^{k-1} (\Delta_{jh+1} t) [(\Delta_{jh+1} t)^{-1} \| V(j, h, h+1)\, U(j, 0, h)\, x - U(j, h, h+1)\, U(j, 0, h)\, x \|].$$

Consistency guarantees for $x$ in $N_0'$ and any $\epsilon > 0$, that when $j$ is sufficiently large

$$(\Delta_{jh+1} t)^{-1} \| V(j, h, h+1)\, U(j, 0, h)\, x - U(j, h, h+1)\, U(j, 0, h)\, x \| \leqslant \frac{\epsilon}{L(x)\, T}.$$

Then

$$\| V(j, 0, k)\, x - U(j, 0, k)\, x \| \leqslant L(x) \sum_{h=0}^{k-1} \Delta_{jh+1} t \left( \frac{\epsilon}{L(x)\, T} \right) \leqslant \epsilon \frac{t_{jk}}{T} \leqslant \epsilon.$$

Corollary A. *If in addition to the hypotheses of this theorem, $U$ is a CFL-stable approximation to itself at each $x$ in $N_0$ (that is, $U(j, 0, k)$ is continuous at each $x$ in $N_0$ uniformly over $j$, $k$ obeying $0 \leqslant j$, $0 \leqslant k \leqslant k(j)$), then*

$$\| V(j, 0, k)\, x - U(j, 0, k)\, x \|$$

*converges to zero as $j \to \infty$ for all $x$ in $N_0$.*

Proof. This corollary follows from the inequality

$$\| V(j, 0, k)\, x - U(j, 0, k)\, x \| \leqslant \| V(j, 0, k)\, x - V(j, 0, k)\, x' \| + \| V(j, 0, k)\, x' - U(j, 0, k)\, x' \| + \| U(j, 0, k)\, x' - U(j, 0, k)\, x \|.$$

Let $x'$ be chosen in $N_0'$. The first and third terms of the upper bound may be made arbitrarily small simply by choosing $x'$ close to $x$, since $U$ and $V$ are continuous at $x$ uniformly over $j$. The middle term may be made small by taking $j$ large.

If the mapping $U$ depends only on the initial and terminal times, as would a solution operator of a well-posed differential equation initial value problem, then $U$ may be represented by $U^{\#}(\tau, t)$ obeying $U(j, h, k) = U^{\#}(t_{jh}, t_{jk})$ and we assert

COROLLARY B. *If in addition to the hypotheses of Theorem 1, $U$ may be represented by a $U^{\#}$ as above such that $U^{\#}(0, t)$ is continuous in $t$ and continuous as a map of $N_0$ uniformly on $0 \leqslant t \leqslant T$, then for all $x$ in $N_0$*

$$\| V(j, 0, k)\, x - U^{\#}(0, t')\, x \|$$

*converges to zero as $j \to \infty$.*

A proof very nearly the same as that for Corollary A may be constructed using the inequality

$$\begin{aligned} \| V(j, 0, k)\, x - U^{\#}(0, t')\, x \| &\leqslant \| V(j, 0, k)\, x - V(j, 0, k)\, x' \| \\ &+ \| V(j, 0, k)\, x' - U^{\#}(0, t_{jk})\, x' \| + \| U^{\#}(0, t_{jk})\, x' - U^{\#}(0, t_{jk})\, x \| \\ &+ \| U^{\#}(0, t_{jk})\, x - U^{\#}(0, t')\, x \| . \end{aligned}$$

## 5. ON STABILITY CRITERIA

The following is an immediate corollary of the Taylor expansion theorem to be found in Dieudonné [8]:

THEOREM. *Let $E$, $F$ be Banach spaces and $S$ be a neighborhood of $\{e_1 + \eta e_2 \mid 0 \leqslant \eta \leqslant 1\}$, where $e_1$, $e_2$ are in $E$. If $f$ is a $p$ times continuously differentiable[1] mapping of $S$ into $F$, then*

$$\begin{aligned} &\| f(e_1 + e_2) - f(e_1) \| \\ &\leqslant \sum_{j=1}^{p-1} \frac{1}{j!} \| e_2 \|^j \| f^{(j)}(e_1) \| + \frac{1}{p!} \| e_2 \|^p \sup_{0 \leqslant \eta \leqslant 1} \| f^{(p)}(e_1 + \eta e_2) \| . \end{aligned}$$

Now using the notation of the previous section, we allow $V$ such continuous differentiability as may be called for. We want a condition which will imply

[1] The derivative $f^{(1)}$ of mapping $f$ is defined as follows. For each $e_1$ in $E$, $f^{(1)}(e_1)$ is the map of $E$ to $F$ such that for all $e_2$

$$f^{(1)}(e_1)\, e_2 = \lim_{\alpha \to 0} \frac{1}{\alpha} [f(e_1 + \alpha e_2) - f(e_1)].$$

Under suitable conditions of continuous differentiability, $f^{(1)}(e_1)$ may be shown to be a linear map of $E$. We assume such for $V$. Higher order derivatives are defined in an analogous way [8].

the Lipschitzian condition (1) at each $x$ in $N_0$. Note that we must now require $X$ to be a Banach space. Applying the theorem above, we note (taking $p = 1$) that

$$\| V(j, h, k)\, x_1 - V(j, h, k)\, x_2 \| \leqslant \| x_1 - x_2 \| \sup_{0 \leqslant \xi \leqslant 1} \| [V(j, h, k)]^{(1)} (\xi x_1 + (1 - \xi)\, x_2) \| .$$

Thus we would seek some guarantee that for each $x$ in $N_0$ and all $x_1$ within some neighborhood of $x_2 = U(j, 0, h)\, x$, this supremum has a finite upper bound independent of $j, h, k$.

Recalling that

$$V(j, h, k) = V(j, k-1, k)\, V(j, k-2, k-1) \cdots V(j, h, h+1) = \prod_{i=1}^{k-h} V(j, k-i, k-i+1),$$

we use a chain rule to convert this supremum into

$$\lambda = \sup_{0 \leqslant \xi \leqslant 1} \left\| \prod_{i=1}^{k-h} \{[V(j, h+i-1, h+i)]^{(1)}\, V(j, h, h+i-1) \times (\xi x_1 + (1-\xi)\, x_2)\} \right\|$$

Since $[V(j, h+i-1, h+i)]^{(1)}\, V(j, h, h+i-1)\,(\xi x_1 + (1-\xi)\, x_2)$ is a linear mapping of $X$, we can observe that as in the linear case, given consistency, a sufficient condition for convergence is the uniform boundedness of a family of products of linear operators. This is the essence of most linearized stability analysis of nonlinear difference schemes. Moreover, this points out the theoretical virtue of such pragmatically virtuous practices and also the theoretical vice of not (for obvious reasons) demanding such boundedness on neighborhoods of $x_2$'s of the exact trajectories $U(j, 0, h)\, x$, $x$ in $N_0$. An interesting discussion related to this point is found in Richards, Lanning, and Torrey [9].

If a constant $K$ can be found so that the terms of the product

$$\prod_{i=1}^{k-h} \| [V(j, h+i-1, h+i)]^{(1)}\, V(j, h, h+i-1)\,(\xi x_1 + (1-\xi)\, x_2) \|$$

are $\leqslant 1 + (\Delta_{jh+i} t)\, K$ when $\| x_1 - x_2 \|$ is sufficiently small, then

$$\lambda \leqslant \prod_{i=1}^{k-h} (1 + (\Delta_{jh+i} t)\, K) \leqslant e^{K(t_{jk} - t_{jh})} \leqslant e^{KT},$$

a bound independent of $j, h, k$.

The theorem quoted above may be used for $p$ other than one to obtain sums of norms for which bounds may be demanded. These will involve higher derivatives of $V$ and in only one term involve a supremum. However, all will require the exact trajectories for $x_2$.

## 6. Concluding Remarks

An interpretation of the proof given for Theorem 1 is illustrated in figure two. The distance between

$$V(j, h, k)\, U(j, 0, h)\, x \qquad \text{and} \qquad V(j, h+1, k)\, U(j, 0, h+1)\, x$$

must be $0(\Delta_{jh+1}t)$ uniformly over $x$, the initial point in $N_0'$, over $j \geqslant 0$, the index of mesh size, and over $h$, the index specifying the step at which the difference scheme is first applied, $0 \leqslant h \leqslant k(j)$.

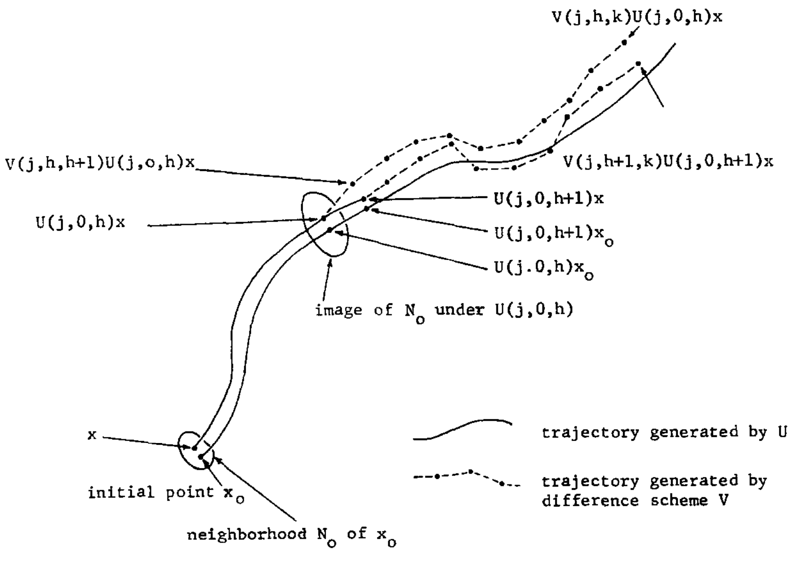


Fig. 2.

The space chosen as $X$ will generally be determined by the problem whose solution is to be approximated and the class of solutions which is to be considered. The consistency condition is met typically on some normed space $A$ of sufficiently regular functions, in say $(R^n)^{R^m}$. Often the natural choice for $X$ is a completion of $A$. The extensions $f^e$ (notation of section three) of approximating (discrete) $f$ are typically in $A$.

This work suffers from the same limitations (see Birkhoff [10]) as the original Lax theory in that prescriptions for choosing the correct space $X$, for extending discrete functions to the continuum, for constructing stable difference methods and for determining rates of convergence are not given. A beginning on these problems for linear partial differential equations has been made by Birkhoff and Varga [11].

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